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Gravitational F-terms of $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)$ gauge theories

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Abstract: We use the generalized Konishi anomaly equations and R-symmetry anomaly to compute the exact perturbative and non-perturbative gravitational F-terms of four-dimensional $\mathcal{N}=1$ supersymmetric gauge theories. We formulate the general procedure for computation and consider chiral and non-chiral $\mathrm{SU}(N)$ gauge theories.

Keywords: Nonperturbative Effects, Anomalies in Field and String Theories, Supersymmetric Effective Theories.

Contents

1. Introduction

An exact computation of gravitational F-terms of $\mathcal{N}=1$ supersymmetric gauge theories coupled to $\mathcal{N}=1$ supergravity in four dimensions is of much theoretical importance. Recently, Dijkgraaf and Vafa suggested a matrix model description where the gravitational F-terms can be computed by summing up the non-planar matrix diagrams [1]. The assumption made is that the relevant fields are the glueball superfields S_i and the F-terms are holomorphic couplings of the glueball superfields to gravity. The DV matrix proposal has been proven diagrammatically in [2, 3].

In this paper we consider gravitational F-terms of the form

$$\Gamma_G = \int d^4x d^2\theta G_{\alpha\beta\gamma} G^{\alpha\beta\gamma} F_1(S_i)\,, \tag{1.1}$$

where $G_{\alpha\beta\gamma}$ is the $\mathcal{N}=1$ Weyl superfield. According to the DV proposal, $F_1(S_i)$ is the partition function of the corresponding matrix model evaluated by summing the genus one diagrams with S_i being the 't Hooft parameter. In a non-trivial graviphoton background

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these F-terms belong to a series of corrections given by

$$\hat{\Gamma}_G = \sum_{g=1}^{\infty} g \int d^4x d^2\theta G_{\alpha\beta\gamma} G^{\alpha\beta\gamma} (F_{\delta\zeta} F^{\delta\zeta})^{g-1} F_g(S_i) \,, \tag{1.2}$$

where now $F_{\delta\zeta}$ is the graviphoton field strength and the $F_g(S_i)$ are the genus g contributions to the matrix model partition function. When the graviphoton field strength vanishes, only the $g = 1$ term in (1.2) remains.

In this paper we will compute such gravitational F-terms in a trivial graviphoton background for chiral and non-chiral supersymmetric gauge theories. The matrix model approach to gravitational F-terms has been recently studied in [4, 5]. The approach we will take is to use the generalized Konishi anomaly equations. Such a technique has been applied in [6] to gauge theory with matter in the adjoint representation of U(N) and extended to the case with a non-trivial graviphoton background in [7]. The gauge groups SO / Sp with various matter contents were considered in [8]. In the models that we will consider the matter fields transform in the fundamental and antisymmetric representations of the gauge groups, and we will be able to perform an exact computation.

We will formulate the general procedure for computation and consider chiral and non-chiral SU(N_c) gauge theories. In the analysis we will need to consider the perturbative and non-perturbative F-terms in the glueball superfields. The perturbative series in the glueball superfields will be computed by solving a set of differential equations with respect to the tree level couplings and using the generalized Konishi anomaly equations. An important ingredient in the analysis is the gravitationally deformed chiral ring of the gauge theory. The computation of the non-perturbative terms will require the use of the U$(1)_R$ anomaly.

The paper is organized as follows. In section 2 we discuss the classical chiral ring of $\mathcal{N} = 1$ supersymmetric gauge theories and its deformation when coupled to $\mathcal{N} = 1$ supergravity. We then consider the classical and quantum generalized Konishi relations in the supergravity background. In section 3 we present the procedure for the computation of the F-terms. The analysis has two parts. One part is the computation of the perturbative series in the glueball superfield. This can be done by solving a set of differential equations with respect to the tree level couplings while using the generalized Konishi anomaly equations. The second part is the computation of the non-perturbative terms in the glueball superfield. Here we will use the requirement that the U$(1)_R$ anomaly has to be reproduced correctly. In section 4 we consider $\mathcal{N} = 1$ super Yang-Mills with non-chiral matter in the fundamental representation and a tree level superpotential. We will compute the exact F-terms both in un-Higgsed and Higgsed branches of the theory. We will then consider the relation of the computation to the vector model [9]. In section 5 we will perform a similar analysis with chiral matter in the fundamental and antisymmetric representations. In appendices A and B we provide details of computations for the non-chiral and chiral matter models, respectively.

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2. The deformed chiral ring

In this section we will discuss the classical chiral ring of $\mathcal{N} = 1$ supersymmetric gauge theories and its deformation when coupled to $\mathcal{N} = 1$ supergravity. We will then consider the classical and quantum generalized Konishi relations in the supergravity background.

2.1 Classical chiral ring

Consider first an $\mathcal{N} = 1$ supersymmetric gauge theory in flat space with a gauge group G and some matter supermultiplets. We will denote the four-dimensional Weyl spinor supersymmetry generators by Q_α and $\bar{Q}_{\dot{\alpha}}$. Chiral operators are operators annihilated by $\bar{Q}_{\dot{\alpha}}$. For instance, the lowest component ϕ of a chiral superfield Φ is a chiral operator. The OPE of two chiral operators is nonsingular and allows for the definition of the product of two chiral operators. The product of chiral operators is also a chiral operator. Furthermore, one can define a ring structure on the set of equivalence classes of chiral operators modulo operators of the form $\{\bar{Q}_{\dot{\alpha}}, \cdots]$.

Denote by V the vector superfield in the adjoint representation of G, by Φ chiral superfields in a representation r of G and by ϕ their lowest component. The field strength (spinor) superfield is $W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V$ and is a chiral superfield. One has

$$\{W_\alpha^{(r)}, W_\beta^{(r)}\} = 0\,, \qquad W_\alpha^{(r)}\phi^{(r)} = 0\,, \tag{2.1}$$

modulo $\{\bar{Q}_{\dot{\alpha}}, \cdots]$ terms, where we noted that ϕ transforms in a representation r of the gauge group G such that $W_\alpha^{(r)} = W_\alpha^a T^a(r)$ with $T^a(r)$ being the generators of the gauge group G in the representation r.

Consider next the coupling of the supersymmetric gauge theory to a background $\mathcal{N} = 1$ supergravity. We denote by $G^{\alpha\beta\gamma}$ the $\mathcal{N} = 1$ Weyl superfield. In the following we will denote by W_α the supersymmetric field strength as well as its lowest component, the gaugino, and similarly for $G^{\alpha\beta\gamma}$. The chiral ring relations (2.1) are deformed to

$$\{W_\alpha^{(r)}, W_\beta^{(r)}\} = 2G_{\alpha\beta\gamma}W^{\gamma(r)}\,, \qquad W_\alpha^{(r)}\phi^{(r)} = 0\,. \tag{2.2}$$

Together with Bianchi identities of $N = 1$ supergravity these relations generate all the relations in the deformed chiral ring. Some relations that will be used later are [6]

$$\begin{aligned} [W^2, W_\alpha] &= 0\,, \qquad W^2 W_\alpha = -\frac{1}{3}G^2 W_\alpha\,, \qquad W^2 W^2 = -\frac{1}{3}G^2 W^2\,, \\ G^4 &= (G^2)^2 = 0\,. \end{aligned} \tag{2.3}$$

Throughout the paper we will follow the conventions used in [6].

In addition to the above kinematical relations one has kinematical relations for the matter fields and dynamical relations from the variation of the tree level superpotential W_{tree}

$$\phi \frac{\partial W_{\text{tree}}}{\partial \phi} = 0\,. \tag{2.4}$$

2.2 Konishi anomaly relations

The classical chiral ring relations are, in general, modified quantum mechanically. The classical relations arising from (2.4) have a natural generalization as anomalous Ward identities of the quantized matter sector in a classical gauge(ino) and supergravity background. The classical Konishi equation reads

$$\bar{D}^2 J = \phi' \frac{\partial W_{\text{tree}}}{\partial \phi} , \tag{2.5}$$

where J is the generalized Konishi current and $\delta\phi = \phi'(\phi)$ is the generalized Konishi transformation. This relation gets an anomalous contribution in the quantum theory. It takes the form [10]–[12]

$$\bar{D}^2 J = \phi_i' \frac{\partial W_{\text{tree}}}{\partial \phi_i} + \frac{1}{32\pi^2} \left(W_{\alpha i}{}^j W^\alpha{}_j{}^k + \frac{1}{3} G^2 \delta_i^k \right) \frac{\partial \phi_k'}{\partial \phi_i} , \tag{2.6}$$

where i, j and k are gauge indices and their contraction is in the appropriate representation.

Since the divergence $\bar{D}^2 J$ is $\bar{Q}$-exact it vanishes in a supersymmetric vacuum. Taking the expectation value of (2.6) in a slowly varying gaugino background S we get the Konishi relations in a supergravity background given by G^2

$$\left\langle \phi_i' \frac{\partial W_{\text{tree}}}{\partial \phi} \right\rangle_S + \left\langle \left(\frac{1}{32\pi^2} W_{\alpha i}{}^j W^\alpha{}_j{}^k + \frac{1}{32\pi^2} \frac{G^2}{3} \delta_i^k \right) \frac{\partial \phi_k'}{\partial \phi_i} \right\rangle_S = 0 . \tag{2.7}$$

We will use this relation to determine the supergravity corrections to the chiral correlators, which in turn can be integrated to give the perturbative part of the gravitational F-terms of the corresponding $\mathcal{N} = 1$ gauge theory.

In the next section we will discuss the gravitational F-terms in more detail and we will outline the methods used to calculate the perturbative part of the gravitational couplings. We will also propose a non-perturbative completion of the gravitational couplings analogous to the non-perturbative completion of the effective superpotential by the Veneziano-Yankielowicz potential [13]. The proposal will be based on $\mathrm{U}(1)_R$ anomaly considerations.

3. Gravitational F-terms of $\mathcal{N} = 1$ gauge theories

We are interested in the low energy description of a four dimensional $\mathcal{N} = 1$ supersymmetric gauge theory in the background of $\mathcal{N} = 1$ supergravity. The assumption is that the relevant field is the glueball superfield S and the F-terms are holomorphic couplings of the glueball superfield to gravity.

In the absence of supergravity, the only relevant F-term is the effective glueball superpotential

$$\Gamma_0 = \int d^4x d^2\theta W_{\text{eff}}(S) , \tag{3.1}$$

where

$$S = -\frac{1}{32\pi^2} \operatorname{Tr} W_\alpha W^\alpha . \tag{3.2}$$

In the matrix model description Γ_0 is computed by summing up planar diagrams and adding a non-perturbative Veneziano-Yankielowicz superpotential.

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When coupled to supergravity there is a gravitational F-term of the form

$$\Gamma_1 = \int d^4x d^2\theta W_1(S) G^2 \,. \tag{3.3}$$

In the matrix model description it is computed by summing up non-planar diagrams and adding a non-perturbative contribution. Note that terms with higher powers of G vanish due to the chiral ring relation (2.3).

Our aim in this paper will be to determine the S dependence of the gravitational couplings (3.3). We will consider chiral and non-chiral $\mathrm{SU}(N_c)$ gauge theories with matter in the fundamental and antisymmetric representations of the gauge group.

3.1 Computation of $W_1(S)$

Consider the supersymmetric gauge theory with a tree level superpotential

$$W_{\text{tree}} = \sum_I g_I \sigma_I \,, \tag{3.4}$$

where σ_I are gauge invariant chiral operators and g_I the tree level couplings. The gradient equations for the holomorphic part of the effective action read

$$\frac{\partial \left(W_{\text{eff}} + G^2 W_1 \right)}{\partial g_I} = \langle \sigma_I \rangle_S \,. \tag{3.5}$$

The expectation values are taken in a slowly varying (classical) gaugino and gravitino background.

As first discussed in [10], for a gauge theory in the absence of a supergravity background the Konishi relations (2.7) can be used to solve for the expectation values $\langle \sigma_I \rangle_S$ as a function of S and the tree level couplings. One can then integrate (3.5) to determine the dependence of W_{eff} on the tree level couplings. For the gravitational coupling $W_1(S)$ a similar reasoning applies. However, we will have to take into account the effects of the supergravity background on the correlators of chiral operators [6].

The perturbative part of $W_1(S)$. In the absence of gravity the correlators of chiral operators factorize

$$\langle \sigma_I \sigma_J \rangle = \langle \sigma_I \rangle \langle \sigma_J \rangle \,. \tag{3.6}$$

In the matrix model description this is the feature of the planar limit. Here and in some of the equations in the following we omit for simplicity the subscript S. Eq. (3.6) can be used in the relations (2.7) in order to solve for $\langle \sigma_I \rangle_S$ as a function of (S, g_I).

However, in the presence of supergravity the chiral correlators do not factorize (non-planarity in the matrix model description) and instead we have

$$\langle \sigma_I \sigma_J \rangle = \langle \sigma_I \sigma_J \rangle_c + \langle \sigma_I \rangle \langle \sigma_J \rangle \,, \tag{3.7}$$

with analogous relations for correlators with more chiral operators. Also, the one point functions have to be expanded in G^2 as

$$\langle \sigma_I \rangle = \langle \sigma_I \rangle_1 + G^2 \langle \sigma_I \rangle_2 \,. \tag{3.8}$$

Note that this expansion is exact in the chiral ring due to the fact that G^4 vanishes modulo $\bar{D}$ exact terms. Thus, we have to express $\langle \sigma_I \rangle_1$ and $\langle \sigma_I \rangle_2$ as functions of S and g_I.

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In the next section we will show explicitly that there are enough relations (2.7) to solve for $\langle\sigma_I\rangle, \langle\sigma_J\rangle$ as well as for the connected correlators $\langle\sigma_I\sigma_J\rangle_c$. The perturbative part of the gravitational coupling $W_1(S)$ is then obtained by integrating the gravitational contribution $\langle\sigma_K\rangle_2$ in (3.8) for the $\langle\sigma_K\rangle$ appearing in the tree level potential (3.4), with respect to the couplings g_K.

A crucial ingredient in the analysis is the fact that connected correlators of three or more chiral operators vanish in the gravitationally deformed chiral ring. Let us show this property diagrammatically in the case of massive matter. For the chiral model the diagrammatic argument is less straightforward and we will invoke a different one.

Connected correlators vs. G^2. A generic n-point function can first be expanded via its connected parts, due to its Feynman-graph interpretation

$$\langle\sigma_I\sigma_J\sigma_K\ldots\rangle = \langle\sigma_I\sigma_J\sigma_K\ldots\rangle_c + \langle\sigma_I\rangle\langle\sigma_J\sigma_K\ldots\rangle_c + \cdots + \langle\sigma_I\rangle\langle\sigma_J\rangle\langle\sigma_K\rangle\ldots. \tag{3.9}$$

We will show that the connected n-point function can be estimated by

$$\langle\sigma_{I_1}\sigma_{I_2}\ldots\sigma_{I_n}\rangle_c \sim G^{2(n-1)}. \tag{3.10}$$

Recall that already G^4 vanishes in the chiral ring.

The proof is done perturbatively in the couplings g_K. First note that the order in G^2 goes up with the genus of the diagram, such that the lowest power in G^2 comes from planar diagrams. Second, all planar diagrams can be generated from a planar one loop diagram, by inserting vertices on its lines and connecting them (in a planar way). Operations like this do not change the order of G^2 of a diagram. Thus to estimate the order of G^2 of a given correlator it suffices to estimate the simplest one loop diagram.

Note that in this way the interaction terms of the theory do not enter the estimate. The power of G^2 is determined by the genus of a diagram and the number of insertions in a correlator.

We consider in the following non-chiral matter in the fundamental representation, $(\tilde{Q}_i^a, Q_a^i)$. $a = 1,\ldots,N_c$ are color indices and $i,j = 1,\ldots,N_f$ the flavor indices. The meson is defined by $M_i^j = \tilde{Q}_i^a Q_a^j$. Matter in the adjoint has been analysed in [6]. We estimate the lowest order contribution in the couplings expansion. The diagram is a circle with n legs running to the n insertions $M(x_i)$. At the leg of $M(x_i)$ the momentum p_i is leaving the circle. We have

$$\begin{aligned}
&\langle M(x_1)M(x_2)\cdots M(x_n)\rangle_c \sim \\
&\sim (n-1)!\int\prod_{i=1}^{n}\left(\frac{d^4p_i d^2\pi_i}{(2\pi)^4}e^{-ip_ix_i}\right)\times\delta^4\left(\sum_{i=1}^{n}p_i\right)\delta^2\left(\sum_{i=1}^{n}\pi_i\right)\int\prod_{i=1}^{n}ds_i\int d^4k d^2\kappa\,\times \\
&\quad\times\operatorname{tr}\left(e^{\left[-s_1((k-p_1)^2+W(\kappa-\pi_1)+m)-s_2((k-p_1-p_2)^2+W(\kappa-\pi_1-\pi_2+m)-\cdots-s_n(k^2+W\kappa+m)\right]}\right),
\end{aligned}$$

where we used the a Schwinger form of the matter propagator. Integrating over p_i and k, while putting $x_i = x_j$, we remain with the fermionic integrations

$$\frac{1}{(2\pi)^{2n}}\frac{1}{2^{2n}}\int\prod_{i=1}^{n}\left(\frac{ds_i e^{-s_i m}}{s_i^2}\right)\int\prod_{i=2}^{n}(d^2\pi_i)d^2\kappa \times$$
$$\times\,\mathrm{tr}\left[\left(1 - s_1 W(\kappa+\pi_2+\cdots+\pi_n) + \frac{(s_1 W(\kappa+\pi_2+\cdots+\pi_n))^2}{2}\right)\cdots\right.$$
$$\left.\cdots\left(1 - s_n W\kappa + \frac{(s_n W\kappa)^2}{2}\right)\right]. \tag{3.11}$$

The fermionic integrals lead to a cancelation of the parameters s_i terms in the denominator and we get

$$\frac{1}{(2\pi)^{2n}}\frac{1}{2^{2n}}\left(\frac{W^2}{2}\right)^n\int\prod_{i=1}^{n}\left(\frac{ds_i e^{-s_i m}}{s_i^2}\right)\prod_{i=1}^{n}(s_i^2) = \frac{1}{(2\pi)^{2n}}\frac{1}{2^{3n}}\left(\frac{W^2}{m}\right)^n. \tag{3.12}$$

Using the ring relations then gives

$$\langle M(0)^n\rangle_c \sim \frac{(n-1)!}{(2\pi)^{2n}}\frac{1}{2^{3n}3^{n-1}m^n}\,\mathrm{tr}(W^2)G^{2(n-1)}\,. \tag{3.13}$$

Nonperturbative part of $W_1(S)$ via anomaly. The procedure outlined above determines $W_1(S)$ up to an integration constant independent of the couplings g_I. We will argue below that this integration constant can be determined by the one loop exact $\mathrm{U}(1)_R$ anomaly. We will fix the integration constant for generic effective superpotentials by matching to the one calculated below, in the limit that all couplings except for the mass are sent to zero.

Consider an $\mathrm{SU}(N_c)$ $\mathcal{N}=1$ supersymmetric gauge theory with n_F chiral multiplets in the fundamental representation. In our notations, $\mathrm{SU}(N_c)$ super Yang-Mills with N_f flavors has $n_F = 2N_f$. The index of the fundamental representation is $C(fund) = 1$.[1]

In our conventions the anomaly $\mathcal{A}$ of the R-symmetry is given by (see e.g. [14, 15, 16])

$$\mathcal{A} = -\frac{1}{3}G^2\left[N_c^2 - 1 - \frac{n_F N_c}{3}\right] + S\left[2N_c - \frac{n_F}{3}\right]. \tag{3.14}$$

Note that here and in the following whenever we write G^2 we will assume it to be normalized appropriately $G^2 \to G^2/32\pi^2$. Also, contributions to the anomaly from gravitinos in the loop are not included, since we treat gravity as a background. It is easy to see from (3.14) that the gauginos have R-charge equal to one and the chiral multiplets have R-charge equal to $2/3$ such that their fermionic components have R-charge equal to $-1/3$.

The effective potential including the gravitational coupling that reproduces the anomaly (3.14) is given by

$$\Gamma(S,G^2) = N_c S\left[-\log\frac{S}{\Lambda^3}+1\right] - \frac{n_F}{2}S\log\frac{\Lambda}{m} + \frac{1}{6}G^2\left[(N_c^2-1)\log\frac{S}{\Lambda_0^3} + n_F N_c\log\frac{\Lambda_0}{m}\right]. \tag{3.15}$$

[1] Here we find it convenient not to distinguish between fundamental and antifundamental representations.

Explicitly, the transformations of the potential (3.15) under the $\mathrm{U}(1)_R$ with

$$S(x,\theta,\bar{\theta}) \to e^{-2i\alpha} S(x, e^{i\alpha}\theta, e^{-i\alpha}\bar{\theta})\,, \tag{3.16}$$

and

$$G^2(x,\theta,\bar{\theta}) \to e^{-2i\alpha} G^2(x, e^{i\alpha}\theta, e^{-i\alpha}\bar{\theta})\,, \tag{3.17}$$

reproduce the Yang-Mills fields contribution to the anomaly (3.14)

$$\delta_{\mathrm{U}(1)_R}\Gamma(S,G^2) = -\frac{1}{3}(N_c^2-1)G^2 + 2N_c S\,. \tag{3.18}$$

The dependence on the scales of the effective potential cannot be fixed by the anomalous transformation properties. The generalized Konishi anomaly relations will constrain the scale dependence. In (3.15) there are four places where the scales Λ or Λ_0 can appear. The generalized Konishi anomaly requires that the scales of each term at a given order in the gravitational expansion in G^2 have to be the same. We will discuss this in more detail in section 4.3.

4. $\mathrm{SU}(N_c)$ with fundamental matter

In this section will consider $\mathrm{SU}(N_c)$ $\mathcal{N}=1$ supersymmetric gauge theory with matter in the fundamental representation $(\tilde{Q}_i^a, Q_a^i)$ and with a tree level superpotential

$$W_{\mathrm{tree}}(M) = m\,\mathrm{tr}(M) + \lambda\,\mathrm{tr}(M^2)\,, \tag{4.1}$$

with the meson $M_i^j = \tilde{Q}_i^a Q_a^j$. Here $a = 1,\ldots,N_c$ are color indices and $i,j = 1,\ldots,N_f$ the flavor indices.

For this model we will calculate the the gravitational F-term. We will consider first the case $N_f = 1$ and then the general N_f. We will then compare the analysis to the vector model calculation.

4.1 Generalized Konishi equations

Consider first the case $N_f = 1$. The theory with a tree level superpotential given by (4.1) has two classical vacua at $M = 0$ and $M = -m/2\lambda$. Classically, in the first vacuum the gauge group is unbroken, in the second it is Higgsed to $\mathrm{SU}(N_c-1)$.

The Konishi variations $\delta Q_i = Q_i$ and $\delta Q_i = Q_i M$ lead to the equations

$$\begin{aligned} S &= m\langle M\rangle + 2\lambda\langle M^2\rangle + \frac{N_c}{3}G^2\,,\\ S\langle M\rangle &= m\langle M^2\rangle + 2\lambda\langle M^3\rangle + \frac{N_c+1}{3}\langle M\rangle G^2\,, \end{aligned} \tag{4.2}$$

respectively. We note that S and G^2 are treated as background fields such that we need not worry about connected correlators including either S or G^2. If the correlators $\langle M^n\rangle$ factorized, the above relations would force G^2 to vanish, as expected. In a supergravity

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background we have to expand the correlators in powers of G^2, as discussed in section 3.1 and use the ring relations (2.3) to reduce the number of unknowns. Let us do that explicitly.

The expansions of the correlators of M are

$$\begin{aligned} \langle MM \rangle &= \langle MM \rangle_c + \langle M \rangle \langle M \rangle , \\ \langle MMM \rangle &= 3 \langle MM \rangle_c \langle M \rangle + \langle M \rangle^3 , \end{aligned} \tag{4.3}$$

with

$$\begin{aligned} \langle M \rangle &= M_0 + M_1 G^2 , \\ \left\langle M^2 \right\rangle_c &\sim G^2 , \qquad \left\langle M^3 \right\rangle_c \sim G^4 . \end{aligned} \tag{4.4}$$

The unknowns are M_0, M_1 and $\langle M^2 \rangle_c$. Inserting the above expansions in (4.2) leads to equations at order $O(G^0)$ and order $O(G^2)$. They are solved by

$$\begin{aligned} M_0 &= \frac{-m \pm \sqrt{m^2 + 8\,S\,\lambda}}{4\,\lambda} , \\ M_1 &= \frac{-m \pm (1 - 2\,N_c)\,\sqrt{m^2 + 8\,S\,\lambda}}{6\,(m^2 + 8\,S\,\lambda)} , \\ \left\langle M^2 \right\rangle_c &= \frac{m^2 + 8\,S\,\lambda \mp m\,\sqrt{m^2 + 8\,S\,\lambda}}{-12\,m^2\,\lambda - 96\,S\,\lambda^2} . \end{aligned} \tag{4.5}$$

The two branches of the square root in (4.5) correspond to the two classical vacua, with $M = 0$ and $M = -\frac{m}{2\lambda}$. This can be seen by using the $S \to 0$ limit. Note that the fact that the connected part of the cubic correlator $\langle M^3 \rangle_c$ vanishes in the chiral ring has been used by us in solving (4.2).

Considering the more general variations $\delta Q_i = Q_i M^k$ one can derive the relations

$$S \left\langle M^k \right\rangle = m \left\langle M^{k+1} \right\rangle + 2\lambda \left\langle M^{k+2} \right\rangle + \frac{N_c + k}{3} \left\langle M^k \right\rangle G^2 . \tag{4.6}$$

It is straightforward to check that these equations are solved by M_0, M_1 and $\langle M^2 \rangle_c$ given by (4.5) with $\langle M^n \rangle_c = 0$ for $n \geq 3$.

4.2 Gravitational F-term

Using (3.5) and the form of the tree level superpotential (4.1) we get for the effective superpotential W_{eff} and the gravitational F-term W_1 the differential equations

$$\begin{aligned} \frac{\partial W_{\text{eff}}}{\partial m} &= \langle M \rangle |_{O(G^0)} , \qquad & \frac{\partial W_{\text{eff}}}{\partial \lambda} &= \left\langle M^2 \right\rangle |_{O(G^0)} , \\ \frac{\partial W_1}{\partial m} &= \langle M \rangle |_{O(G^2)} , \qquad & \frac{\partial W_1}{\partial \lambda} &= \left\langle M^2 \right\rangle |_{O(G^2)} . \end{aligned} \tag{4.7}$$

Inserting the expectation values (4.5) we get differential equations that determine the dependence of the effective superpotential and the gravitational F-term on the tree level couplings.

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The differential equations at order $O(G^0)$ have been integrated in [17] and give the effective superpotential

$$W_{\text{eff}} = N_c S\left(-\log\frac{S}{\Lambda^3}+1\right)+S\log\frac{m}{\Lambda}-\frac{S}{2}+S\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8S\lambda}{m^2}}\right)- \\ -\frac{m^2}{8\lambda}+\frac{m^2}{8\lambda}\sqrt{1+\frac{8\lambda S}{m^2}}\,, \tag{4.8}$$

where the integration constant has been fixed by matching to the Veneziano-Yankielowicz potential. This means that W_{eff} reproduces the first line of (3.15) in the limit $\lambda\to 0$. Similarly, the equations at order $O(G^2)$ can be integrated to give W_1

$$W_1 = \frac{1}{3}\left(-N\log m-\frac{2N-1}{2}\log\left(1+\sqrt{1+\frac{8S\lambda}{m^2}}\right)-\frac{1}{4}\log\left(1+\frac{8S\lambda}{m^2}\right)\right)+C(S,G^2)\,,$$

where the integration constant $C(S,G^2)$ is independent of the tree level couplings. $C(S,G^2)$ is fixed by taking the limit $\lambda\to 0$ and requiring that W_1 reproduces the second line of (3.15).

With this matching the complete form of W_1 reads

$$W_1 = \frac{1}{3}\left(\frac{N_c^2-1}{2}\log\left(\frac{S}{\Lambda_0^3}\right)-N_c\log\frac{m}{\Lambda_0}--\frac{2N_c-1}{2}\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8S\lambda}{m^2}}\right)- \\ -\frac{1}{4}\log\left(1+\frac{8S\lambda}{m^2}\right)\right). \tag{4.9}$$

SU(N_c) with N_f flavors. To determine the gravitational couplings for the model with N_f fundamental flavors around the classical vacuum with $N_f^- = N_f - N_f^+$ Higgsing quarks is only slightly more complicated. The details of this computation are given in appendix A.

We recall first the result for W_{eff} which has been computed in [17]. W_{eff} reads

$$W_{\text{eff}} = N_c S\left(-\log\frac{S}{\hat\Lambda^3}+1\right)-N_f\left(\frac{S}{2}+\frac{m^2}{8\lambda}\right)+(N_f^+-N_f^-)\frac{m^2}{8\lambda}\sqrt{1+\frac{8\lambda}{m^2}S}+ \\ +S\log\left[\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\lambda}{m^2}S}\right)^{N_f^+}\left(\frac{1}{2}-\frac{1}{2}\sqrt{1+\frac{8\lambda}{m^2}S}\right)^{N_f^-}\right]. \tag{4.10}$$

The scales involved are the UV scale Λ, the scale $\hat\Lambda$ for the theory with N_f massive quarks around $Q=0$, and the scale $\tilde\Lambda$ for the theory with N_f^+ massive quarks around $Q=0$ and N_f^- Higgsing quarks,

$$\hat\Lambda^{3N_c} = \Lambda^{3N_c-N_f}m^{N_f} = \tilde\Lambda^{3(N_c-N_f^-)}\left(\frac{m^2}{2\lambda}\right)^{N_f^-}. \tag{4.11}$$

The order $O(G^2)$ contributions to the relevant condensates in this vacuum are given by

$$\langle \mathrm{tr}\, M\rangle|_{O(G^2)} = N_f^+M_1^+ + N_f^-M_1^-\,, \tag{4.12}$$

and

$$\langle \mathrm{tr}\, M^2 \rangle\big|_{O(G^2)} = 2\left(N_f^+ M_0^+ M_1^+ + N_f^- M_0^- M_1^-\right) + \left(N_f^{+2} B^+ + N_f^{-2} B^-\right) , \tag{4.13}$$

where the M_0, M_1, B with superscript $+(-)$ refer to the solutions of the Konishi relations (A.5) on the un-Higgsed (Higgsed) branch given in the appendix.[2] Integrating with respect to m and λ and fixing the integration constant in the way outlined above gives the gravitational F-term

$$\begin{aligned} W_1 = & \frac{1}{6}\left[\left(N_c - N_f^-\right)^2 - 1\right] \log \frac{S}{\Lambda_0^3} - \frac{1}{6}\left[N_f^- N_f^- - 2N_c N_f^-\right] \log \frac{m}{-2\lambda\Lambda_0^2} - \\ & - \frac{1}{6}\left[N_f^- N_f^- + 2N_c N_f^+\right] \log \frac{m}{\Lambda_0} - \frac{N_f^{+2} + N_f^{-2}}{12} \log\left(1 + \frac{8\lambda S}{m^2}\right) + \\ & + \frac{1}{6}\left(N_f^+ - N_f^-\right)(N_f - 2N_c) \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8\lambda S}{m^2}}\right). \end{aligned} \tag{4.14}$$

4.3 Parameter space

We now want to point out in what sense the perturbative part of W_1 gives us a consistency check on our approach to determine the integration constant from the $\mathrm{U}(1)_R$ anomaly. As an illustrative example we will consider the classical vacuum with $N_f^- = 0$ Higgsing quarks and $N_f^+ = N_f$ which corresponds to the vacuum with unbroken gauge group. In the full quantum solution (4.14) this classical vacuum is connected in the parameter space to the classical vacuum with $N_f^- = N_f$ Higgsing quarks where the gauge group is maximally Higgsed from $\mathrm{SU}(N_c)$ to $\mathrm{SU}(N_c - N_f)$.

By passing through the branch cut of the square root we can go from one semiclassical region to the other. If the integration constant is indeed determined by the $\mathrm{U}(1)_R$ anomaly of the classically unbroken gauge group, which accounts for the strong coupling dynamics, then changing the branch should provide the appropriate terms to adjust the integration constant correctly. In other words for the vacuum with unbroken gauge group we expect in the limit $\lambda \to 0$ the gravitational F-term to be

$$W_1^{uh} = \frac{1}{3}\left(\frac{N_c^2 - 1}{2} \log\left(\frac{S}{\Lambda_0^3}\right) - N_f N_c \log \frac{m}{\Lambda_0}\right).$$

On the other hand for the maximally Higgsed vacuum we expect

$$W_1^h = \frac{1}{3}\left(\frac{(N_c - N_f)^2 - 1}{2} \log\left(\frac{S}{\Lambda_0^3}\right) + \left[N_f^2 - 2N_c N_f\right] \log\left(\frac{\Lambda_0}{m_h}\right) + \frac{N_f^2}{2} \log\left(\frac{\Lambda_0}{m}\right)\right).$$

The physical meaning of this potential is, that it comes from integrating out, first the massive W-bosons at the Higgsing scale $m_h = \sqrt{-m/2\lambda}$ and then the remaining uncharged matter at the scale m. As usual the dependence on the mass scales is determined by integrating the Konishi relations but the important thing is that changing the branch of

[2] See also [18] for a detailed analysis.

the square root in (4.14) provides exactly the necessary terms to change the coefficient of the $\log S$ term according to

$$\left(N_c^2 - 1\right) \log S \to \left((N_c - N_f)^2 - 1\right) \log S\,, \tag{4.15}$$

which is precisely the expected behavior. From the general solution (4.14) it is easy to see that the same is true for the general vacuum with $N_f^- \neq 0$ Higgsing quarks.

Remarks on the energy scales. We have seen that the functional dependence of the holomorphic couplings on S can be fixed by anomaly matching. However, the scales Λ_0 appearing in the potential remain unfixed. More precisely, neither the Konishi nor the $\mathrm{U}(1)_R$ anomaly restricts the scale in the $\log(S/\Lambda_0^3)$ to be the same as the scale in the $\log(m/\Lambda_0)$ term.

However, the complete solution fixes the two to be the same. This can be seen as follows. The complete solution allows to continue the gravitational coupling W_1 from the un-Higgsed vacuum to the Higgsed vacuum. This analytic continuation mixes the scale Λ_0 that saturates S with the ones that saturate m and m_h. Thus, these scales have to be the same.

4.4 Comparison to the vector model

The anomaly equation (4.2) can also be derived from the zero dimensional vector model

$$F(\hat{N}, N_f, g_s; m, \lambda) = -\ln \int dQ d\tilde{Q} e^{-\frac{1}{g_s} W_{\mathrm{tree}}(\tilde{Q}Q)}\,. \tag{4.16}$$

with $W_{\mathrm{tree}}(\tilde{Q}Q) = mM + \lambda M^2$. The Ward identities for the variation $\delta Q = Q$ and $\delta Q = QM$ are

$$\begin{aligned} g_s \hat{N} &= m \langle M \rangle + \lambda \left\langle M^2 \right\rangle, \\ g_s \hat{N} \ \langle M \rangle + g_s \langle M \rangle &= m \left\langle M^2 \right\rangle + \lambda \left\langle M^3 \right\rangle. \end{aligned} \tag{4.17}$$

Making the identification $S - \frac{N_c}{3} G^2 = g_s \hat{N}$ and $g_s = -\frac{1}{3} G^2$, one reproduces the Konishi anomaly equations (4.2). Here $\hat{N}$ is the size of the vectors in the vector model.

Planar and non-planar diagrams. According to the proposal of Dijkgraaf and Vafa [1] the gravitational coupling $W_1(S)$ should be given by diagrams of genus one in a corresponding matrix (vector) model. It was noted in [3] that string theory arguments imply that there are also planar contributions to $W_1(S)$ which vanish at the extrema of $W_{\mathrm{eff}}(S)$. It has been shown in [6] that in the adjoint model the planar contributions to W_1 can be generated from W_{eff} by shifting S. This is not precisely the case in our model.

Consider the case $N_f = 1$ with the gravitational F-term given by eq. (4.9). We can identify the planar contributions to W_1 by shifting $S \to S - \frac{1}{3} N_c G^2$.[3] This shift amounts to replacing one S by G^2 in each of the planar diagrams and the factor of N_c accounts for

[3] The shift depends on the matter representation, this is why in our case the shift differs from the shift in the adjoint model [6].

the index loop without an S. The appropriate combinatorial factor which is induced by this procedure (for each diagram we now have to choose an additional index loop to be free of S) is taken care of by the derivative with respect to S. In this sense we have

$$W_1^{\text{planar}} = \frac{\partial W_{\text{eff}}}{\partial S}\delta S\,, \tag{4.18}$$

with $\delta S = -\frac{N_c}{3}G^2$.

Consider the perturbative part of W_{eff} (4.8). Shifting S gives:

$$-\frac{N_c}{3}\log\frac{m}{\Lambda} - \frac{N_c}{3}\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\lambda S}{m^2}}\right). \tag{4.19}$$

We can identify this part in W_1 with the identification of the scales $\Lambda_0 = \Lambda$. The perturbative non-planar part of W_1 is then

$$-\frac{1}{12}\log\left(1+\frac{8\lambda S}{m^2}\right) + \frac{1}{6}\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8\lambda S}{m^2}}\right). \tag{4.20}$$

However, it is easy to see that the shift of S in the non-perturbative part of W_{eff} does not reproduce the non-perturbative planar part of W_1 that goes like N_c^2.

5. A chiral SU(6) model

We consider in this section the chiral SU(6) model analysed in [17]. The matter content of this model consists of two antifundamental flavors $\bar{Q}_i^I$ and one flavor X^{ij} in the antisymmetric representation. Capital $I, J = 1, 2$ denote flavor indices and i, j denote color indices. The relevant gauge invariant operators are

$$T = \varepsilon_{IJ}\bar{Q}_i^I\bar{Q}_j^J X^{ij}\,, \tag{5.1}$$

and

$$U = PfX = X^{i_1j_1}X^{i_2j_2}X^{i_3j_3}\varepsilon_{i_1j_1i_2j_2i_3j_3}\,. \tag{5.2}$$

We will consider the tree level superpotential

$$W_{\text{tree}} = hT + gU + \lambda TU\,, \tag{5.3}$$

where, as explained in detail in [17], the term λTU will provide us with a region of the quantum parameter space where, despite the absence of a massive vacuum, we can reliably determine the integration constant by matching the $\mathrm{U}(1)_R$ anomaly.

Again, we have to take the effect of the supergravity background into account. This leads to the following general expansion rules for the chiral correlators

$$\begin{aligned}\langle A\rangle &\equiv A_0 + G^2 A_1\,,\\ \langle AB\rangle &\equiv \langle A\rangle\langle B\rangle + G^2\langle AB\rangle_c\,.\end{aligned} \tag{5.4}$$

Concerning the three point correlators we will argue that the connected part vanishes in the chiral ring as in the cases of SYM with non-chiral matter in the fundamental representation and the adjoint model [6]. The diagrammatic proof seems to be more intricate in this case and we will instead use the Konishi equations to show perturbatively in λ (or S) that the connected part $\langle ABC\rangle_c$ of any three point correlator vanishes. We can then find a closed system of equations which allows to solve for $T_{0,1}, U_{0,1}$ and $\langle TU\rangle_c$ in order to determine the dependence of the gravitational F-terms on the tree level couplings.

5.1 Connected n-point correlators

Our strategy will be the following. We will start by deriving the most general Konishi relations based on the variations

$$\delta A = AT^n U^m \,, \tag{5.5}$$

where A stands for either $\bar{Q}$ or X. We will assume that the connected parts of all the n-point functions for $n \geq 3$ have nonvanishing contributions at order $O(G^2)$. It is then easy to verify that for $n+m>1$ the Konishi relations can generate a closed system of equations for the correlators. However, these equations turn out to be linearly dependent and we will resort to the following perturbative argument. First, we are going to show by induction that for $\lambda \to 0$ the connected parts of the n-point correlators on the un-Higgsed branch vanish for $n > 2$. On dimensional grounds we then have

$$\langle T^n U^m \rangle_c = \lambda S^{n+m} f(\lambda S) \,, \tag{5.6}$$

where in the limit $\lambda \to 0$ the function f has to go with some positive (or zero) power of λ. We then turn on λ. Due to the linear dependence in this case we can only solve in terms of one unknown, say, T_1 defined in the expansions (5.4). By requiring that the correlator $\langle T^a U^b\rangle$ show the behavior of (5.6) we can solve for T_1 up to $(a+b-1)$'th order in λ. We find perfect agreement with the expansion of the complete solution which we obtain when we assume that the connected three point correlators $\langle ABC\rangle_c$ vanish.

The model with $\lambda \to 0$. If we set $\lambda \to 0$ we can solve explicitly for $T_{0,1}, U_{0,1}$ and $\langle TU\rangle_c, \langle TT\rangle_c, \langle UU\rangle_c$. The solution is given by

$$\begin{aligned} T_0 &= \frac{S}{h} & T_1 &= -\frac{2}{h}\,, \\ U_0 &= \frac{S}{g} & U_1 &= -\frac{1}{g}\,, \\ \langle TT\rangle_c &= -\frac{S}{3h^2}\,, & \langle TU\rangle_c &= 0\,, \\ \langle UU\rangle_c &= -\frac{S}{3g^2}\,. \end{aligned} \tag{5.7}$$

The variations (5.5) lead to

$$h\left\langle T^{n+1}U^m\right\rangle = S\left\langle T^n U^m\right\rangle - \frac{G^2}{3}(n+6)\left\langle T^n U^m\right\rangle , \tag{5.8}$$

for $\delta\bar{Q}^1_i = \bar{Q}^1_i T^n U^m$ and

$$h\left\langle T^{n+1}U^m\right\rangle + 3g\left\langle T^nU^{m+1}\right\rangle = 4S\left\langle T^nU^m\right\rangle - \frac{G^2}{3}(n+3m+15)\left\langle T^nU^m\right\rangle, \qquad (5.9)$$

for $\delta X^{ab} = X^{ab}T^nU^m$. Using the above equations we can verify that the connected correlators $\langle T^nU^m\rangle_c$ vanish for $n+m>2$. First, plugging the solutions (5.7) into the equations (5.8) and (5.9) and assuming that all the connected correlators up to $\langle T^nU^m\rangle_c$ for $n+m$ fixed vanish we find that the correlators $\langle T^{n+1}U^m\rangle_c$ and $\langle T^nU^{m+1}\rangle_c$ also vanish. Verifying by explicit calculation that all the connected three point correlators vanish finally allows us to conclude that also all higher connected correlators vanish. For finite λ dimensional analysis then restricts the correlators to be of the form given in (5.6).

The model with finite λ. For a finite value of λ the Konishi relations (5.8), (5.9) get modified to

$$\begin{aligned} h\left\langle T^{n+1}U^m\right\rangle + \lambda\left\langle T^{n+1}U^{m+1}\right\rangle &= S\left\langle T^nU^m\right\rangle - \frac{G^2}{3}(n+6)\left\langle T^nU^m\right\rangle, \\ h\left\langle T^{n+1}U^m\right\rangle + 3g\left\langle T^nU^{m+1}\right\rangle + 4\lambda\left\langle T^{n+1}U^{m+1}\right\rangle &= 4S\left\langle T^nU^m\right\rangle \\ &\quad -\frac{G^2}{3}(n+3m+15)\left\langle T^nU^m\right\rangle. \end{aligned} \qquad (5.10)$$

If we want to include all the k-point functions we have to consider all equations with $n+m+1 \le k$. The λ term in the Konishi relations also introduces some $k+1$ point functions into the game. The solution of this set of equations will be given in terms of one of the variables which we take to be T_1. Expanding T_1 in a series of λ and imposing on the $k+1$-point functions that they go like λS^{k+1} for small λ determines T_1 up to order k in λ. We have checked explicitly up to order $O(\lambda^8)$. The result we get for T_1 is given by

$$\begin{aligned} T_1 &= -\frac{2}{h} + \frac{3S}{gh^2}\lambda - \frac{28S^2}{3g^2h^3}\lambda^2 + \frac{32S^3}{g^3h^4}\lambda^3 - \frac{344S^4}{3g^4h^5}\lambda^4 + \\ &\quad + \frac{1264S^5}{3g^5h^6}\lambda^5 - \frac{4720S^6}{3g^6h^7}\lambda^6 + \frac{17824S^7}{3g^7h^8}\lambda^7 - \frac{67864S^8}{3g^8h^9}\lambda^8 + O(\lambda^9). \end{aligned} \qquad (5.11)$$

Below we will see that this is in perfect agreement with the complete solution when we take the connected three point correlators to vanish. We expect this argument to be applicable to any order in λ provided one solves (5.10) up to high enough order in $n+m$.

The complete solution. If we assume the connected three point correlators to vanish we can solve for $T_{0,1}, U_{0,1}$ and $\langle TU\rangle_c$ in order to determine the dependence of the gravitational F-terms on the tree level couplings. The details of this calculation including the appropriate anomalous transformations and the corresponding Konishi equations are given in appendix B. We obtain the following results

$$\begin{aligned} T_0 &= -\frac{g}{2\lambda}\left(1 \mp \sqrt{1+\frac{4\lambda}{gh}S}\right), \\ U_0 &= -\frac{h}{2\lambda}\left(1 \mp \sqrt{1+\frac{4\lambda}{gh}S}\right), \end{aligned}$$

$$
\begin{aligned}
T_1 &= -\frac{2}{3h} \mp \frac{4}{3h\sqrt{1+\frac{4\lambda}{gh}S}} + \frac{\lambda S}{3gh^2\left(1+\frac{4\lambda}{gh}S\right)}, \\
U_1 &= \frac{1}{3g} \mp \frac{4}{3g\sqrt{1+\frac{4\lambda}{gh}S}} + \frac{\lambda S}{3g^2h\left(1+\frac{4\lambda}{gh}S\right)}, \\
\langle TU\rangle_c &= \frac{1}{6\lambda}\left(\pm\frac{gh+2\lambda S}{gh\sqrt{1+\frac{4\lambda}{gh}S}}-1\right), \\
\langle TT\rangle_c &= \mp\frac{S}{3h^2\sqrt{1+\frac{4\lambda}{gh}S}}, \\
\langle UU\rangle_c &= \mp\frac{S}{3g^2\sqrt{1+\frac{4\lambda}{gh}S}}, \qquad (5.12)
\end{aligned}
$$

where the (upper) lower sign is valid on the (un)Higgsed branch. Expanding T_1 on the un-Higgsed branch with respect to λ one finds that it agrees with what we have found in (5.11). We consider this a convincing argument that the assumption about the vanishing connected three point correlators is correct.

It was shown in [17] that on the Higgsed branch the theory classically reduces to a pure Sp(2) theory. This fact made it possible to determine the full effective superpotential W_{eff} by matching to the Veneziano-Yankielowicz potential,

$$
\begin{aligned}
W_{\text{eff}} = &-5S\log\frac{S}{\Lambda_6^3}+4S-2S\log 2++S\log gh+\frac{gh}{2\lambda}\left(-1\pm\sqrt{1+\frac{4\lambda}{gh}S}\right)+ \\
&+2S\log\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\frac{4\lambda}{gh}S}\right). \qquad (5.13)
\end{aligned}
$$

Similarly, we will determine here the integration constant for the gravitational F-terms by matching the perturbative solution on the Higgsed branch to the gravitational part of the $\mathrm{U}(1)_R$ anomaly for pure Sp(2). Integrating the order $O(G^2)$ parts of $\langle T\rangle, \langle U\rangle$ and $\langle TU\rangle$, as indicated in (5.4), with respect to h, g and λ yields

$$
W_1^h = -\log gh^2 - \frac{8}{3}\log\frac{gh}{-\lambda\Lambda_0^3} - \frac{1}{12}\log\left(1+\frac{4\lambda}{gh}S\right) + \frac{8}{3}\log\left(1+\sqrt{1+\frac{4\lambda}{gh}S}\right) + C(S). \qquad (5.14)
$$

To determine $C(S)$ we require that in the limit $S\to 0$ the leading $\log S$ term of $W_1(S)$ reproduce the gravitational $\mathrm{U}(1)_R$ anomaly for pure Sp(2). In the conventions we have used throughout the paper this is given by

$$
\frac{1}{6}\dim\left(\mathbf{Adj}_{\mathrm{Sp}(2)}\right)\log S = \frac{5}{3}\log S. \qquad (5.15)
$$

The full solution on the Higgsed branch then reads

$$
W_1^h = \frac{5}{3}\log\frac{S}{\Lambda_0^3} - \log gh^2 - \frac{8}{3}\log\frac{gh}{-\lambda\Lambda_0^3} - \frac{1}{12}\log\left(1+\frac{4\lambda}{gh}S\right) + \frac{8}{3}\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4\lambda}{gh}S}\right). \qquad (5.16)
$$

As usual, the un-Higgsed branch of the parameter space can be reached by passing through the branch cut of the square root in W_1, which leads us to

$$W_1^{uh} = \frac{13}{3}\log\frac{S}{\Lambda_0^3} - \log gh^2 - \frac{1}{12}\log\left(1+\frac{4\lambda}{gh}S\right) - \frac{8}{3}\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4\lambda}{gh}S}\right). \tag{5.17}$$

Expanding the derivative with respect to h of W_1^{uh} as a power series in the coupling λ reproduces (5.11).

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A. Konishi relations SYM with $N_f \neq 1$

The Konishi equations relevant for the perturbation under consideration are

$$\begin{aligned}
\delta Q_i^f &= Q_i^{f'} : & \left(S - \frac{N_c}{3}G^2\right)\delta_f^{f'} &= m\left\langle M_f^{f'}\right\rangle + 2\lambda\left\langle (M^2)_f^{f'}\right\rangle, \\
\delta Q_i^f &= Q_i^{f'} M_h^{h'} : & \left(S - \frac{N_c}{3}G^2\right)\delta_f^{f'}\left\langle M_h^{h'}\right\rangle &= m\left\langle M_f^{f'} M_h^{h'}\right\rangle + 2\lambda\left\langle (M^2)_f^{f'} M_h^{h'}\right\rangle \\
&&& \quad + \frac{1}{3}\delta_f^{h'}\left\langle M_h^{f'}\right\rangle G^2,
\end{aligned} \tag{A.1}$$

We study these equations for the case where the flavor symmetry is not broken. With the conventions

$$\begin{aligned}
\left\langle M_f^{f'}\right\rangle &= \delta_f^{f'} M, \\
\left\langle M_f^{f'} M_h^{h'}\right\rangle &= \left\langle M_f^{f'} M_h^{h'}\right\rangle_c + \left\langle M_f^{f'}\right\rangle\left\langle M_h^{h'}\right\rangle, \\
\left\langle M_f^{f'} M_h^{h'}\right\rangle_c &= \delta_f^{f'}\delta_h^{h'} A + \delta_f^{h'}\delta_h^{f'} B, \\
\left\langle M_f^{f'} M_h^{h'} M_g^{g'}\right\rangle &= \left\langle M_f^{f'} M_h^{h'}\right\rangle_c \left\langle M_g^{g'}\right\rangle + \left\langle M_f^{f'} M_g^{g'}\right\rangle_c \left\langle M_h^{h'}\right\rangle + \\
&\quad + \left\langle M_h^{h'} M_g^{g'}\right\rangle_c \left\langle M_f^{f'}\right\rangle + \left\langle M_f^{f'}\right\rangle\left\langle M_h^{h'}\right\rangle\left\langle M_g^{g'}\right\rangle, \\
\left\langle (M^2)_f^{f'} M_h^{h'}\right\rangle &= \delta_f^{f'}\delta_h^{h'}(3AM + N_f BM + M^3) + \delta_f^{h'}\delta_h^{f'} 2BM .
\end{aligned} \tag{A.2}$$

The Konishi equations then are

$$\begin{aligned}
S &= mM + 2\lambda(M^2 + A + N_f B) + \frac{N}{3}G^2, \\
SM &= m(A + M^2) + 2\lambda\left[3AM + N_f BM + M^3\right] + \frac{N}{3}M\,G^2, \\
0 &= mB + 4\lambda BM + \frac{1}{3}MG^2,
\end{aligned} \tag{A.3}$$

where

$$\begin{aligned} M &= M_0 + M_1 G^2 \,, \\ A &= A_2 G^4 \,, \\ B &= B_1 G^2 \,. \end{aligned} \tag{A.4}$$

In components

$$\begin{aligned} S &= mM_0 + 2\lambda M_0^2 \,, \\ 0 &= mM_1 + 2\lambda(2M_0M_1 + N_f B_1) + \frac{N}{3} \,, \\ 0 &= mB_1 + 4\lambda B_1 M_0 + \frac{M_0}{3} \,, \end{aligned} \tag{A.5}$$

and the redundant equations

$$\begin{aligned} SM_0 &= mM_0^2 + 2\lambda M_0^3 \,, \\ SM_1 &= 2mM_0M_1 + 2\lambda(N_f B_1 M_0 + 3M_0^2 M_1) + \frac{M_0}{3} \,. \end{aligned} \tag{A.6}$$

The solution is given by

$$\begin{aligned} M_0^{\pm} &= \frac{-m \pm \sqrt{m^2 + 8\lambda S}}{4\lambda} \,, \\ M_1^{\pm} &= \mp \frac{N_c}{3\sqrt{m^2+8\lambda S}} + 2\lambda N_f \frac{-m \pm \sqrt{m^2+8\lambda S}}{12\lambda\,(m^2+8\lambda S)} \,, \\ B_1^{\pm} &= \mp \frac{-m \pm \sqrt{m^2+8\lambda S}}{12\lambda\sqrt{m^2+8\lambda S}} \,. \end{aligned} \tag{A.7}$$

B. Details of the chiral model

The transformation

$$\delta \bar{Q}_i^1 = \bar{Q}_i^1 \tag{B.1}$$

gives the chiral ring relation

$$hT + \lambda TU = S - \frac{G^2}{3} N \,, \tag{B.2}$$

for $N = 6$. Here and in the following Konishi relations we will use N and replace it only later by the actual value. If we expand the one point functions in G^2 and take into account the connected correlators we find

$$O(G^0) \;:\; hT_0 + \lambda T_0 U_0 = S \,, \tag{B.3}$$

and

$$O(G^2) \;:\; hT_1 + \lambda\,(T_1U_0 + T_0U_1 + \langle TU\rangle_c) = -\frac{N}{3} \,. \tag{B.4}$$

The chiral ring relation induced by the transformation

$$\delta X^{mn} = X^{mn} \tag{B.5}$$

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is given by

$$hT + 3gU + 4\lambda TU = (N-2)\, S - \frac{G^2}{3}\frac{N\,(N-1)}{2}\,. \tag{B.6}$$

At order $O(G^0)$ and $O(G^2)$ we find respectively

$$\begin{aligned} O(G^0)\ :\ \ hT_0 + 3gU_0 + 4\lambda T_0 U_0 \,&=\, (N-2)\, S\,, \\ O(G^2)\ :\ \ hT_1 + 3gU_1 + 4\lambda\,(T_1U_0 + T_0U_1 + \langle TU\rangle_c) \,&=\, -\frac{N\,(N-1)}{6}\,. \end{aligned} \tag{B.7}$$

From the two equations at order $O(G^2)$ it follows that U_1 and T_1 satisfy

$$gU_1 - hT_1 = 1 \tag{B.8}$$

(for $N = 6$). The variation

$$\delta \bar{Q}^1_i = \bar{Q}^1_i T \tag{B.9}$$

gives rise to

$$hT^2 + \lambda T^2 U = ST - \frac{G^2}{3}\,(N+1)\,T\,, \tag{B.10}$$

which gives the following relevant equation at order $O(G^2)$

$$\begin{aligned} O(G^2):\ \ & 2hT_0T_1 + h\langle TT\rangle_c + \\ & + \lambda\left(T_0^2U_1 + T_0T_1U_0 + T_1T_0U_0 + U_0\langle TT\rangle_c + 2T_0\langle TU\rangle_c\right) + \\ & + \lambda\langle TTU\rangle_c \,=\, ST_1 - (N+1)\,\frac{T_0}{3}\,. \end{aligned} \tag{B.11}$$

Similarly, for the variation

$$\delta \bar{Q}^1_i = \bar{Q}^1_i U \tag{B.12}$$

we find

$$\begin{aligned} O(G^2):\ \ & hT_0U_1 + hT_1U_0 + h\langle TU\rangle_c + \\ & + \lambda\,(2T_0U_0U_1 + T_1U_0U_0 + T_0\langle UU\rangle_c + 2U_0\langle TU\rangle_c) + \\ & + \lambda\langle TUU\rangle_c \,=\mathrm{SU}_1 - \frac{N}{3}U_0\,. \end{aligned} \tag{B.13}$$

Finally,

$$\delta X^{mn} = X^{mn}T \tag{B.14}$$

leads to

$$\begin{aligned} O(G^2):\ \ & 2hT_0T_1 + h\langle TT\rangle_c + 3gT_0U_1 + 3gU_0T_1 + \\ & + 3g\langle TU\rangle_c + 4\lambda\,(T_0T_0U_1 + 2T_0T_1U_0) + \\ & + 8\lambda T_0\langle TU\rangle_c + 4\lambda U_0\langle TT\rangle_c + 4\lambda\langle TTU\rangle_c = (N-2)\,ST_1 - \frac{T_0}{6}\,[N\,(N-1)+2]\,. \end{aligned}$$

We derive another Konishi relation which comes from the variation

$$\delta X^{mn} = X^{mn}U\,. \tag{B.15}$$

The corresponding relation in the chiral ring reads

$$hTU + 3gUU + 4\lambda TUU = (N-2)\,\mathrm{SU} - G^2\frac{U}{6}\left[N\,(N-1)+6\right], \tag{B.16}$$

which at order $O(G^2)$ gives

$$\begin{aligned} O(G^2):\;\; & hT_0U_1 + hT_1U_0 + h\langle TU\rangle_c + 6gU_0U_1 + \\ & + 3g\langle UU\rangle_c + 4\lambda\,(T_1U_0U_0 + 2T_0U_0U_1) + \\ & + 4\lambda T_0\langle UU\rangle_c + 8\lambda U_0\langle TU\rangle_c + 4\lambda\langle TUU\rangle_c = (N-2)\,\mathrm{SU}_1 - \frac{U_0}{6}\left[N\,(N-1)+6\right]. \end{aligned}$$

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